Weak Identification with Many Instruments

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Introduction

- We consider IV (instrumental variable) models with *many weak* instruments
  - Estimation with many instruments
  - How to determine that instruments are weak?
  - Weak identification robust inferences (some new results here)
  - Open questions

Introduction

- **Example 1**: Angrist and Krueger (1991)

$$wage_i = \beta\ education_i + controls + e_i,$$

- Instrument is quarter of birth
- First stage is heterogeneous: law depends on state and birth cohort
- Instruments used: QOB ($\times$ state dummy) ($\times$ year dummy)
  - year of birth (30)
  - year and state of birth (180)
  - year and state of birth, and their interactions (1530)
- Staiger and Stock (1997)- IV may be weak
- Hansen et al. (2008)- instruments are many

Introduction

- **Example 2**: 'Judges design'
- Bhuller, Dahl, Loken and Mogstad (JPE, 2020): "Incarceration, Recidivism, and Employment"

$$recidivism_i = \beta \ \ incarceration_i + controls + e_i,$$

- Instruments: "judge stringency" = the average incarceration rate in other cases a judge has handled
- This is a form of JIVE with instrument-dummies for judge assignment
- Sample size is roughly proportional to the number of judges

Setup

- Linear IV model with one endogenous variable:

$$\begin{cases} Y_i = \beta X_i + (\delta W_i) + e_i \\ X_i = \pi' Z_i + (\gamma W_i) + v_i \end{cases}$$

  where $Z_i \in \mathbb{R}^K$ s.t. $\mathbb{E}[e_i|Z_i, W_i] = \mathbb{E}[v_i|Z_i, W_i] = 0$
- Data is i.i.d., $i = 1, ..., N$
- Many instruments: $K \to \infty$ as $N \to \infty$ (up to $K = \lambda N$)
- Weak instruments: π is small in some sense
- For most results errors are heteroskedastic

Outline

1. Estimation
2. Weak Identification: detection
3. Weak IV robust inferences
   - AR test
   - Other tests: LM
4. Adding covariates
5. Conclusions and Open questions

Overview

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Setup

- Assume away covariates (we will add them in the last section)

$$\begin{cases} Y_i = \beta X_i + e_i \\ X_i = \pi' Z_i + v_i \end{cases}$$

  where $Z_i \in \mathbb{R}^K$ s.t. $\mathbb{E}[e_i|Z_i] = \mathbb{E}[v_i|Z_i] = 0$
- Data is i.i.d., $i = 1, ..., N$
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TSLS

Most commonly known estimator is Two-Stage Least Squares (TSLS)

- First stage- finding the optimal instrument = best predictor

$$\widehat{X}_i = \widehat{\pi} Z_i$$

- Second stage: estimate structural equation using $\widehat{X}_i$ as the instruments
- Optimal instrument under homoskedasticity: $\mathbb{E}[X_i|Z_i]$ (Chamberlain, 1987)
- Concentration parameter $\frac{\pi' Z' Z \pi}{\sigma_v^2}$ plays as effective sample size (Stock and Yogo, 2005)

Estimation with Many IV

- First stage: $X_i = \pi' Z_i + v_i$
- If many regressors in the first stage, they might 'overfit' the noise
- Estimated optimal instrument is endogenous $\mathbb{E}[\widehat{X}_i e_i] \neq 0$
- For homoscedastic TSLS: $\widehat{X}_i = \pi' Z_i + v' Z (Z'Z)^{-1} Z_i$

$$\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N} \widehat{X}_i e_i\right] = K\sigma_{ev}$$

- Endogeneity is growing in K, leads to bias
- Bias of the IV estimator increases with the number of moment conditions/instruments (Bekker, 1994, Newey and Smith, 2004)

Estimation with Many IV

Suggestions on how to remove endogeneity:

- Sample splitting (Angrist and Krueger, 1995):
  - split sample to halves
  - select/estimate optimal instrument on one half
  - estimate β on the other half
- Jackknife (Angrist et al., 1999)
  - estimate optimal instrument for observation i on sample excluding i
  - use estimated optimal instrument

Estimation with Many IV

- $\widehat{\beta}_{TSLS} - \beta = \frac{X' P_Z e}{X' P_Z X}$, where $P_Z = Z(Z'Z)^{-1}Z'$
- Bias comes from $\mathbb{E}[X' P_Z e] = \mathbb{E}[v' P_Z e] = \sum_i P_{ii} \mathbb{E}[v_i e_i]$ the diagonal of the projection matrix, $trace(P_Z) = K$
- Idea: remove the diagonal

$$\widehat{\beta}_{JIV} = \frac{\sum_{i \neq j} X_i P_{ij} Y_j}{\sum_{i \neq j} X_i P_{ij} X_j}$$

- It is very close to jackknife (numerical differences are tiny)
- Diagonal removal can be done to many estimators: JIVE-LIML and JIVE-Fuller (Hausman et al., 2012), JIVE-ridge (Hansen and Kozbur, 2014)

Estimation with Many IV

$$\begin{cases} Y_i = \beta X_i + e_i, \\ X_i = \pi' Z_i + v_i, \end{cases}$$

- TSLS is consistent when $\frac{\pi' Z' Z \pi}{K} \to \infty$ (Chao and Swanson, 2005)
- When $\frac{\pi' Z' Z \pi}{\sqrt{K}} \to \infty$, JIVE, JIVE-Fuller and JIVE-LIML are consistent (Hausman et al, 2012)
- When $\frac{\pi' Z' Z \pi}{\sqrt{K}} \to \infty$, JIVE, JIVE-Fuller and JIVE-LIML are asymptotically gaussian
  - Wald confidence sets and t-statistics can be used
  - Estimation of standard errors is non-trivial (Hausman et al, 2012)

Estimation with Many IV: Summary

- Many instruments can be hurtful if they do not extract additional information from the first stage
- Over-fitting creates a bias
- One should avoid using TSLS with many instruments
- Jack-knifing or diagonal removal is very fruitful idea

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What is Weak Identification?

- If $\frac{\pi' Z' Z \pi}{\sqrt{K}} \to \infty$, then JIVE or JIVE-LIML are consistent and asymptotically gaussian
- What if there are better estimators (work well for weaker cases)?
  - Negative statement: in the best possible scenario – only π and β are unknown, if $\frac{\pi' Z' Z \pi}{\sqrt{K}} \asymp const$, there exists no asymptotically consistent robust test (Mikusheva and Sun, 2022)
- How to know in practice if $\frac{\pi' Z' Z \pi}{\sqrt{K}}$ is large enough to trust Wald confidence sets?

What is Weak Identification?

- Mikusheva and Sun (2022): pre-test for weak identification
  - Size distortions of JIVE-Wald depend on $\frac{\mu^2}{\Upsilon\sqrt{K}}$, where $\mu^2 = \sum_{i \neq j} P_{ij}(\pi' Z_i)(\pi' Z_j)$ and Υ- measure of the first stage uncertainty
  - Derived a cut-off, if $\frac{\mu^2}{\Upsilon\sqrt{K}}$ is above cut-off, then JIVE- Wald test has small size distortion
  - Have estimator $\widetilde{F}$ for $\frac{\mu^2}{\Upsilon\sqrt{K}}$
  - Got a cut-off for $\widetilde{F}$

Weak Identification: detection

- Our pre-test is based on the empirical measure:

$$\widetilde{F} = \frac{1}{\sqrt{K}\sqrt{\widehat{\Upsilon}}} \sum_{i=1}^{N} \sum_{j \neq i} P_{ij} X_i X_j,$$

  here $\widehat{\Upsilon}$ is an estimate of uncertainty in the first stage
- If $\widetilde{F} > 4.14$, then the JIVE- Wald test has less than 10 % size distortion
- Suggestion: if $\widetilde{F}$ is low, one should use "robust" tests
- Stata package implementing pre-test and robust tests: `manyweakiv` (beta version)

Re-visiting Angrist and Krueger (1991)

- Research question: returns to education. Y_i is the log weekly wage, X_i is education
- Instruments: quarter of birth. Justification is related to compulsory education laws:
  - 180 instruments: 30 quarter and year of birth interactions (QOB-YOB) and 150 quarter and state of birth interactions (QOB-POB)
  - 1530 instruments: full interactions among QOB-YOB-POB
- The sample contains 329,509 men born 1930-39 from the 1980 census
- This paper sparked the weak IV literature. It is a running example for multiple papers

Re-visiting Angrist and Krueger (1991)

	FF	$\widetilde{F}$	JIVE-Wald	Robust AR	Robust LM
180 instruments	2.4	13.4	[0.066,0.132]	[0.008,0.201]	[0.067,0.135]
1530 instruments	1.3	6.2	[0.024,0.121]	[-0.047, 0.202]	[0.022,0.127]

Table: Angrist and Krueger (1991) Pre-test Results

Notes: Results on pre-tests for weak identification and confidence sets for IV specification underlying Table VII Column (6) of Angrist and Krueger (1991). The confidence sets are constructed via analytical test inversion.

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Weak IV-Robust Tests: Refresher, Fixed K

- $Y_i = \beta X_i + e_i$, Z_i-instrument ($\mathbb{E}[e_i|Z_i] = 0$)
- $H_0 : \beta = \beta_0$. Define $e(\beta_0) = Y - \beta_0 X$
- AR (Anderson-Rubin) statistics:

$$e(\beta_0)' Z \Sigma^{-1} Z' e(\beta_0) \sim \chi^2_K$$

  Σ is a covariance matrix of $e'Z$ or a good estimate of it
- Size is robust to weak IV

What Changes with $K \to \infty$?

- Homoskedastic AR statistics for fixed K:

$$\frac{1}{\sigma^2} e(\beta_0)' Z (Z'Z)^{-1} Z' e(\beta_0) \sim \chi^2_K$$

- χ^2_K is a diverging distribution for large K
- $e(\beta_0)' P_Z e(\beta_0)$ has a non-zero mean $\mathbb{E} e' P_Z e = \sum_{i=1}^N P_{ii} \mathbb{E} e_i^2$
- Idea: remove the diagonal $\sum_{i \neq j} e_i(\beta_0) P_{ij} e_j(\beta_0)$
- Use CLT for quadratic forms (U-statistics)

AR test with many instruments

- The infeasible leave-one-out AR is

$$AR_0(\beta_0) = \frac{1}{\sqrt{K\Phi_0}} \sum_{i \neq j} e_i(\beta_0) P_{ij} e_j(\beta_0),$$

  for $\Phi_0 = \frac{2}{K} \sum_{i \neq j} P_{ij}^2 \sigma_i^2 \sigma_j^2$
- Under $H_0 : \beta = \beta_0$ we have $AR_0(\beta_0) \Rightarrow N(0,1)$
- Need $K \to \infty$ for asymptotic distribution
- Rejects for large values of AR
- Need to estimate the variance

AR: Variance Estimation

$$\Phi_0 = \frac{2}{K} \sum_{i \neq j} P_{ij}^2 \sigma_i^2 \sigma_j^2$$

- Idea 1 (Crudu et al, 2021): $\widehat{\sigma}_i^2 = e_i^2(\beta_0)$
- It gives correct size, robust toward heteroscedasticity, but power is problematic at distant alternatives
- Residualizing $e(\beta_0)$ with respect to Z ($M = I - P_Z$)

$$\widehat{\sigma}_i^2 = \left(M_i \mathbf{e}(\beta_0)\right)^2, \mathbb{E}[\widehat{\sigma}_i^2] \neq \sigma_i^2$$

- "Cross-fit" variance estimator (Newey et al, 2018; Kline et al., 2020):

$$\widehat{\sigma}_i^2 = \frac{1}{1 - P_{ii}} e_i(\beta_0) M_i e(\beta_0)$$

AR: Variance Estimation

$$\Phi_0 = \frac{2}{K} \sum_{i \neq j} P_{ij}^2 \sigma_i^2 \sigma_j^2$$

- Use proxy $\widehat{\sigma}_i^2 = \frac{1}{1-P_{ii}} e_i(\beta_0) M_i e(\beta_0)$
- Challenge is that we need a double sum:

$$\mathbb{E}\left[(e_i M_i e)(e_j M_j e)\right] = (M_{ii} M_{jj} + M_{ij}^2)\sigma_i^2 \sigma_j^2$$

- Our suggested estimator (Mikusheva and Sun, 2022):

$$\widehat{\Phi}_2 = \frac{2}{K} \sum_{i \neq j} \frac{P_{ij}^2}{M_{ii} M_{jj} + M_{ij}^2} \left[e_i(\beta_0) M_i e(\beta_0)\right] \left[e_j(\beta_0) M_j e(\beta_0)\right]$$

AR: Variance Estimation

$$\Phi_0 = \frac{2}{K} \sum_{i \neq j} P_{ij}^2 \sigma_i^2 \sigma_j^2$$

- Anatolyev and Sølvsten (2020): one may get unbiased proxy for $\sigma_i^2 \sigma_j^2$ by using "leave-three-out" estimator

$$\widehat{\sigma_i^2 \sigma_j^2} = e_i(\beta_0) e_j(\beta_0) \sum_k \tilde{M}_{ik,-(ij)} e_k(\beta_0) [e_j(\beta_0) - Z_j' \widehat{\delta}_{-(ijk)}]$$

  where $\widehat{\delta}_{-(ijk)}$ is OLS from regressing $e(\beta_0)$ on Z leaving three observations (i, j, k) out
- There are explicit formulas for "leave-(one/two/three)-out" available, but numerical complexity increases

Feasible AR

$AR(\beta_0) = \frac{1}{\sqrt{K\widehat{\Phi}}} \sum_{i \neq j} e_i(\beta_0) P_{ij} e_j(\beta_0)$ rejects when $AR(\beta_0) > z_{1-\alpha}$

- Where $\hat{\Phi}$ can be:
  - Crudu et al (2021): $\widehat{\Phi}_1 = \frac{2}{K} \sum_{i \neq j} P_{ij}^2 e_i(\beta_0)^2 e_j(\beta_0)^2$
  - Mikusheva and Sun (2022):
    $\widehat{\Phi}_2 = \frac{2}{K} \sum_{i \neq j} \frac{P_{ij}^2}{M_{ii} M_{jj} + M_{ij}^2} [e_i(\beta_0) M_i e(\beta_0)] [e_j(\beta_0) M_j e(\beta_0)]$
  - Anatolyev and Sølvsten: $\widehat{\Phi}_3$ leave-three-out
- All three $\widehat{\Phi}$ are consistent for Φ_0 under the null
- Feasible tests with $\widehat{\Phi}_2$ and $\widehat{\Phi}_3$ achieve the same local power as the infeasible AR
- Feasible tests with $\widehat{\Phi}_2$ and $\widehat{\Phi}_3$ are consistent for distant alternatives
- Variance estimators are ordered in terms of increasing computational complexity

Power of AR

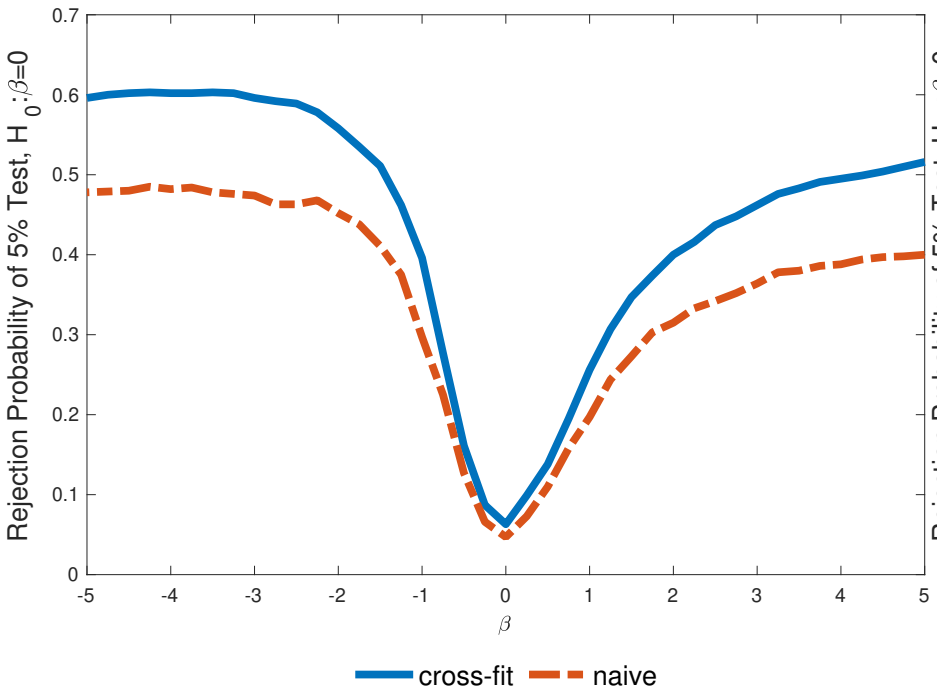
- Under the alternative $\beta = \beta_0 + \Delta$
- Define a leave-one-out information in the sample:

$$\mu^2 = \sum_{i \neq j} P_{ij} (\pi Z_i)(\pi Z_j) \asymp \pi' Z' Z \pi$$

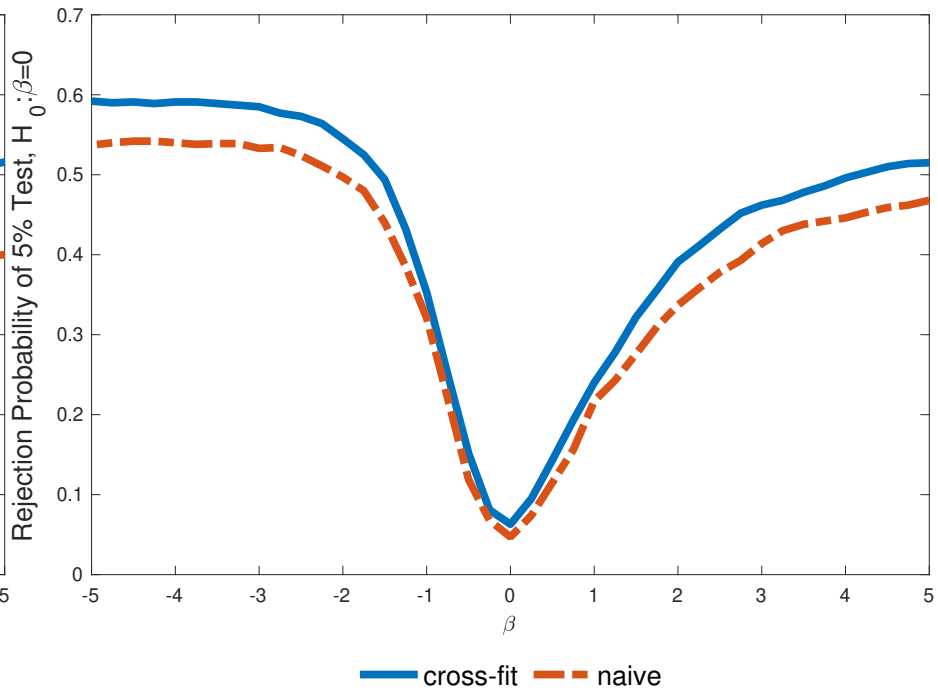
- Power statement: uniformly over a set of local alternative and a (reasonably restricted) set of μ^2 :

$$AR(\beta_0) \Rightarrow \Delta^2 \frac{\mu^2}{\sqrt{K \Phi_0}} + \mathcal{N}(0,1)$$

Variance Estimator is Important



(a) sparse, $\frac{\mu^2}{\sqrt{K}} = 2.5$



(b) dense, $\frac{\mu^2}{\sqrt{K}} = 2.5$

Power curves for leave-one-out AR tests with $\widehat{\Phi}_1$ (red dash) and $\widehat{\Phi}_2$ (blue line) and variance estimators under sparse vs. dense first stage. Instruments are $K = 40$ balanced group indicators, $N = 200$, based on 1,000 simulations

Weak IV-Robust Tests: LM

- Problem: AR is not efficient if identification is strong
- AR uses all instruments "equally"
- LM intends to test a "powerful" combination of instruments $e'Z\pi$,
- Idealistic LM is based on the linear combination $e'(\beta_0)Z\widehat{\pi} = e'(\beta_0)P_Z X$
- Leave-one-out gives us $LM^{1/2} \propto \sum_{i \neq j} e_i(\beta_0)P_{ij}X_j$

Robust LM

- The infeasible leave-one-out LM is

$$LM^{1/2}(\beta_0) = \frac{1}{\sqrt{K\Psi}} \sum_{i \neq j} e_i(\beta_0) P_{ij} X_j,$$

- Under $H_0 : \beta = \beta_0$ we have $LM^{1/2}(\beta_0) \Rightarrow N(0,1)$ as $N, K \to \infty$
- Reject when $\left| LM^{1/2}(\beta_0) \right|$ is large (two-sided test)
- Need an estimator for

$$\Psi = \frac{1}{K} \sum_{i=1}^{N} (\sum_{j \neq i} P_{ij} X_j)^2 \sigma_i^2 + \frac{1}{K} \sum_{i=1}^{N} \sum_{j \neq i} P_{ij}^2 \gamma_i \gamma_j,$$

$\sigma_i^2 = \mathbb{E} e_i^2, \gamma_i = \mathbb{E}[X_i e_i]$

LM: variance estimation

- Matsushita and Otsu (2022):

$$\widehat{\Psi}_1 = \frac{1}{K}\sum_i \widehat{\sigma}_i^2 (\sum_{j\neq i} P_{ij} X_j)^2 + \frac{1}{K}\sum_i \sum_{j\neq i} P_{ij}^2 \widehat{\gamma}_i \widehat{\gamma}_j,$$

  where $\widehat{\sigma}_i^2 = e_i^2(\beta_0)$ and $\widehat{\gamma}_i = X_i e_i(\beta_0)$
- $\widehat{\Psi}_1$ is consistent under the null ($H_0 : \beta = \beta_0$)
- There is a loss of power at alternative: $e(\beta_0) \neq e_i$ and $\widehat{\sigma}_i^2$ overstate the variance

LM: variance estimation (new result)

- We propose to use the ideas of double-cross-fit

$$\widehat{\Psi}_2 = \frac{1}{K}\sum_i \frac{e_i M_i e}{M_{ii}}(\sum_{j\neq i} P_{ij}X_j)^2 + \frac{1}{K}\sum_i\sum_{j\neq i} \widetilde{P}^2_{ij} X_i M_i e X_j M_j e$$

- Here we use
  - $\widehat{\sigma}^2_i = \frac{e_i M_i e}{M_{ii}}$ – unbiased proxy for σ^2_i
  - $\widehat{\gamma}_i = X_i M_i e$ – proxy for γ_i
  - re-weighting $\widetilde{P}^2_{ij} = \frac{P^2_{ij}}{M_{ii}M_{jj}+M^2_{ij}}$ to correct for correlation in proxies

LM: variance estimation (new result)

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- Under assumptions needed for CLT and stricter moment condition, our estimator
  - Consistent under the null (when $\beta = \beta_0$)
  - Consistent for local alternative (when $\beta = \beta_0 + \Delta$ and $\Delta^2 \frac{\mu^2}{K} \to 0$)
  - Inconsistent for global alternatives but still deliver consistent LM test
- Leave-one-out LM with our variance estimation has the same power curves as "infeasible" LM test

Power of LM

- The infeasible leave-one-out LM is

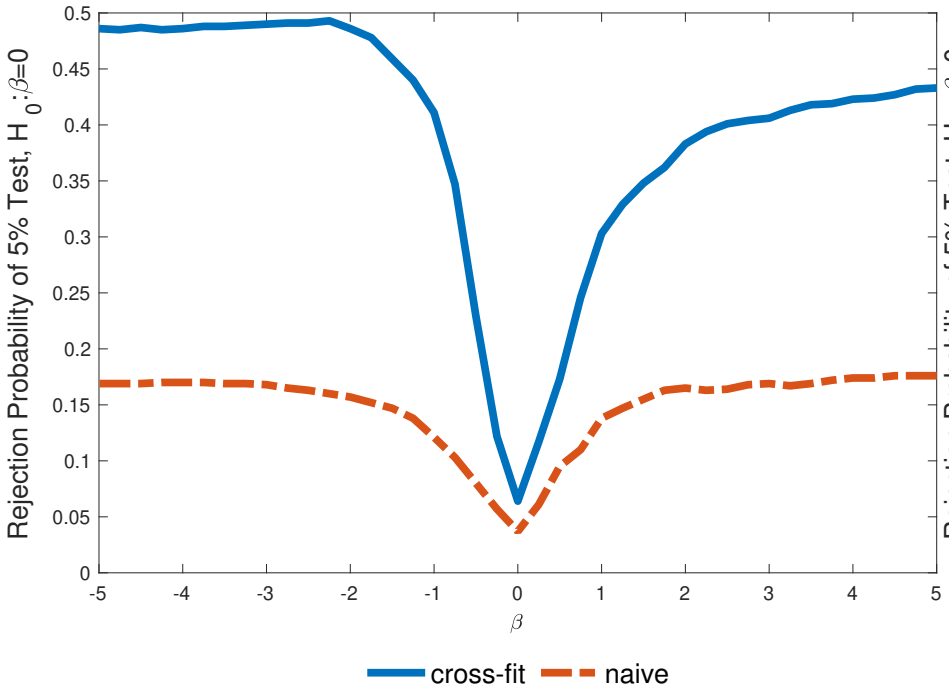
$$LM^{1/2}(\beta_0) = \frac{1}{\sqrt{K\Psi}} \sum_{i \neq j} e_i(\beta_0) P_{ij} X_j$$

- Under the alternative $\beta = \beta_0 + \Delta$, we have $e_i(\beta_0) = Z_i' \pi \Delta + \eta_i$:

$$LM^{1/2} \Rightarrow \Delta \frac{\mu^2}{\sqrt{K\Psi}} + \mathcal{N}(0,1),$$

  uniformly over local alternatives
- LM test has two-sided rejection region
- As soon as $\mu^2/\sqrt{K} \to \infty$, LM is consistent for fixed alternatives

Variance Estimator is Important



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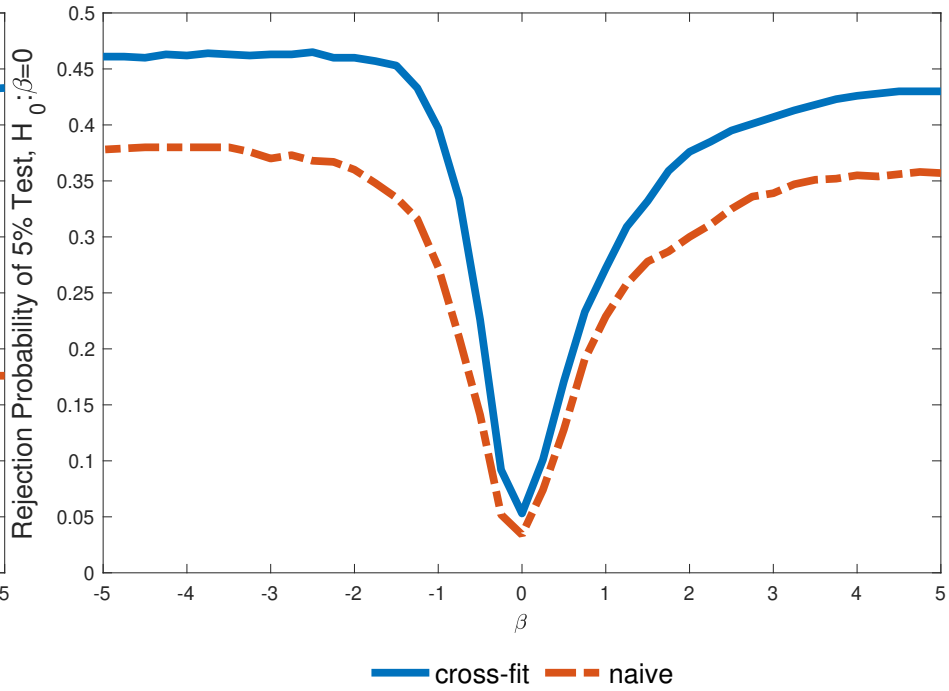

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Power curves for leave-one-out LM with $\widehat{\Psi}_1$ (red dash) and $\widehat{\Psi}_2$ (blue line) variance estimators under sparse vs. dense first stage. Instruments are $K = 40$ balanced group indicators, $N = 200$

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Power Trade-off

- Under the alternative $\beta = \beta_0 + \Delta$, we have :

$$LM^{1/2} \Rightarrow \Delta \frac{\mu^2}{\sqrt{K\Psi}} + \mathcal{N}(0,1),$$

$$AR \Rightarrow \Delta^2 \frac{\mu^2}{\sqrt{K\Phi}} + \mathcal{N}(0,1)$$

- When $\frac{\mu^2}{\sqrt{K}} \to \infty$, AR and LM are asymptotically consistent for fixed alternatives β
- When $\frac{\mu^2}{\sqrt{K}} \to \infty$ but $\frac{\mu^2}{K} \to 0$ local alternatives are:
  - for AR $\{\Delta : \frac{\Delta^2\mu^2}{\sqrt{K}} \leq C\}$ i.e. $|\Delta| \propto \sqrt{\frac{\sqrt{K}}{\mu^2}}$
  - for LM $\{\Delta : \frac{|\Delta|\mu^2}{\sqrt{K}} \leq C\}$ i.e. $|\Delta| \propto \frac{\sqrt{K}}{\mu^2}$
  - AR has slower speed of detection

Conditional Switch Test: CLR

- We may think about combining three statistics optimally

$$\begin{pmatrix} AR(\beta_0) - \Delta^2 \frac{\mu^2}{\sqrt{K\Phi}} \\ LM^{1/2}(\beta_0) - \Delta \frac{\mu^2}{\sqrt{K\Psi}} \\ \widetilde{F} - \frac{\mu^2}{\sqrt{K\Upsilon}} \end{pmatrix} \Rightarrow \mathcal{N}(\mathbf{0}, \Sigma).$$

- AR and LM are for testing β_0 and $\widetilde{F}$ for assessing the strength of identification
- Lim, Wang and Zhang (2022) - suggests an optimal combination test
- Ayyar, Matsushita and Otsu (2022) - suggestions on how to build CLR test

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Adding covariates: what is the problem?

- Linear IV model with one endogenous variable:

$$\begin{cases} Y_i = \beta X_i + \delta W_i + e_i, \\ X_i = \pi' Z_i + \gamma W_i + v_i, \end{cases}$$

  where $Z_i \in \mathbb{R}^K$ s.t. $\mathbb{E}[e_i|Z_i, W_i] = \mathbb{E}[v_i|Z_i, W_i] = 0$

- When there are no covariates (W_i) the bias was removed by removing a diagonal (JIVE)

$$\widehat{\beta}_{JIV} = \frac{\sum_{i \neq j} X_i P_{ij} Y_j}{\sum_{i \neq j} X_i P_{ij} X_j'} = \frac{X' P_{JIV} Y}{X' P_{JIV} X}$$

  we denote $A_{JIV} = \{A_{ij} \cdot 1\{i \neq j\}\}$ for any matrix

- Could we do a similar thing: partial out covariates and remove the diagonal from P_Z?

Adding covariates: what is the problem?

- Let $M_W = I - W(W'W)^{-1}W'$ be partialling out operator
- $Y^\perp = M_W Y$, $X^\perp = M_W X$, $Z^\perp = M_W Z$, $P^\perp = P_{Z^\perp}$,
- Would the following estimator work?

$$\widehat{\beta} = \frac{\sum_{i \neq j} X_i^\perp P_{ij}^\perp Y_j^\perp}{\sum_{i \neq j} X_i^\perp P_{ij}^\perp Y_j^\perp} = \frac{(X^\perp)' P_{JIV}^\perp Y^\perp}{(X^\perp)' P_{JIV}^\perp X^\perp}$$

- No. This is the same as $\widehat{\beta} = \frac{X' M_W P_{JIV}^\perp M_W Y}{X' M_W P_{JIV}^\perp M_W X}$
- Matrix $M_W P_{JIV}^\perp M_W$ has a non-trivial diagonal and produces bias in the estimator

Adding covariates: what is the problem?

- What if we do this in opposite order:

$$\widehat{\beta} = \frac{X'(M_W P^{\perp} M_W)_{JIV} Y}{X'(M_W P^{\perp} M_W)_{JIV} X}$$

- It does not work either

$$(M_W P^{\perp} M_W)_{JIV} W \neq 0$$

  it loses partialling out property

Adding covariates: estimation

- Solution proposed in Chao, Swanson and Woutersen (2023): find $\theta_1, ..., \theta_n$ and diagonal matrix D_θ:

$$M_W(P^\perp - D_\theta)M_W \text{ has zero diagonal}$$

  this problem is linear and solvable for well-balanced designs
- Suggested estimator

$$\widehat{\beta} = \frac{X' M_W (P^\perp - D_\theta) M_W Y}{X' M_W (P^\perp - D_\theta) M_W X}$$

- Chao, Swanson and Woutersen (2023) has proof of consistency and asymptotic gaussianity under some assumptions

Adding covariates: robust inference (new results)

$$\begin{cases} Y_i = \beta X_i + \delta W_i + e_i, \\ X_i = \pi' Z_i + \gamma W_i + v_i, \end{cases}$$

- We can create a weak IV robust test for $H_0 : \beta = \beta_0$ using this idea

$$AR(\beta_0) = \frac{1}{\sqrt{K\Phi}}(Y - \beta_0 X)' M_W (P^{\perp} - D_\theta) M_W (Y - \beta_0 X)$$

- Under the null $AR(\beta_0) \Rightarrow N(0,1)$, reject when $AR(\beta_0)$ is large

Adding covariates: robust inference (new results)

- Denote A_{ij} to be elements of $M_W(P^{\perp} - D_\theta)M_W$, then we can use

$$\widehat{\Phi} = \frac{2}{K}\sum_{i,j}\frac{A_{ij}^2}{M_{ii}M_{jj} + M_{ij}^2}\widehat{\sigma}_i^2\widehat{\sigma}_j^2$$

  where $\widehat{\sigma}_i^2 = \sum_k \frac{M_{ik}}{M_{ii}}(Y_i - \beta_0 X_i)(Y_k - \beta_0 X_k)$, while M_{ij} are elements of $M_{Z,W}$
- The "naive" variance estimator is inconsistent even under the null, since $(Y_i - \beta_0 X_i)^2$ overstates the variance drastically (it has part predictable by W_i)

Overview

1. Estimation
2. Weak Identification: detection
3. Weak IV robust inferences
4. Adding covariates
5. Conclusions and Open questions

Conclusions and Open Questions

- Many instruments come with costs - one needs to find an optimal way to combine them
- Uncertainty about the first stage produces biases of TSLS
- Jackknifing or deleting diagonal is productive idea for both estimation and inference
- The knife-edge case for consistency happens when $\frac{\pi' Z' Z \pi}{\sqrt{K}} \asymp const$
- There is a pre-test for weak identification robust to heteroscedasticity when $K \to \infty$, which depends on the estimator one uses with it
- Robust tests (AR and LM) use the idea of leave-one-out quadratic forms and cross-fit variance estimation
- Adding many covariates is non-trivial

Conclusions and open questions

- Open question: there is a pre-test for whether one can trust JIVE-Wald confidence set/ t-test. JIVE-LIML is more efficient (Hausman et al, 2012), but there is no pre-test for it
- Open question: there is no pre-test that accommodates many covariates either
- Open question: unclear what to do with inferences when there are multiple endogenous variables (sub-vector inference)
- Open question: many instruments framework accommodates well heterogeneous first stage, what to do about heterogeneous structural equation (non-parametric IV)

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